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Complex powers of hypoelliptic pseudodifferential operators

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Abstract

Complex powers of a class of hypoelliptic pseudodifferential operators in $\mathbb{R}^n$, as well as their heat kernels are studied. An application to the Schatten–von Neumann property of pseudodifferential operators is given.

Keywords: Pseudodifferential operators; Weyl–Hörmander calculus; Complex powers; Non-negative operators; Global hypoellipticity

1. Introduction

Complex powers of pseudodifferential operators have been studied by several authors, starting from the works of Seeley [29–31], where the ζ-function $\zeta(z) = \operatorname{Tr} A^z$ for boundary value problems was introduced. Generalizations have been then considered, among others, by Kumano-go and Tsutsumi [17], Kumano-go [16], Richard Beals [2,3], Robert [25], Helffer [12]. Indeed, the study of poles of the zeta function has important applications to index theory, as showed in the celebrated paper by Atiyah, Bott and Patodi [1], and Weyl asymptotics, for which we refer to Duistermaat and Guillemin [9] and also to Shubin [32]. Among other applications, we point out that the study of bounded imaginary powers of pseudodifferential operators also gives informations on the maximal regularity for evolution equations, in view of the theorem of Dore and Venni [8]; see for example Coriasco, Schrohe and Seiler [7] and the references therein.

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Recently the attention has been fixed on complex powers of pseudodifferential operators on manifolds with boundary with a given boundary fibration structure; the operators considered are then elliptic in a calculus which is not, in general, temperate (such as, e.g., the b-calculus [22]). In this context one is interested in the relationships with the geometric properties of the underlying manifold; we refer, for example, to the recent contributions by Schrohe [26,27], Loya [19, 20], Melrose and Nistor [24], and Lauter and Moroianu [18]. In the last two papers the complex powers are used to define various Wodzicki-type residues as generators of the Hochschild cohomology in dimension 0.

In this paper we consider pseudodifferential operators in $\mathbb{R}^n$, whose Weyl symbol belongs to Hörmander's classes $S(m,g)$ associated with a weight function m and a Riemannian metric g, which are temperate and slowly varying; see Hörmander [15, Chapter XVIII]. In this context complex powers have been already considered by Robert [25] for (globally) elliptic symbols diverging at infinity, and consequently with compact resolvent.

Here we treat very general (globally) hypoelliptic operators (see Definition 2.2) whose spectrum may have zero as an accumulation point and whose Weyl symbol is allowed to tend both to zero and to infinity in different directions. For example, the reader may consider the operator

$$-\Delta + \langle x\rangle^{-2\alpha} \quad \text{with } 0 < \alpha < 1$$

(where $\langle x\rangle = (1+|x|^2)^{1/2}$) as a model case. This operator can be regarded as a second order operator in the so-called scattering calculus, which corresponds to the choice $m = \langle\xi\rangle^2$ and $g_{x,\xi} = \langle x\rangle^{-2}|dx|^2 + \langle\xi\rangle^{-2}|d\xi|^2$ (see Melrose [23], and also Schrohe [27] for complex powers of scattering-type operators). However it is not hypoelliptic in that class, in the sense that the derivatives of the symbol weighted according to that metric are not controlled by the symbol itself. Instead, the symbol is hypoelliptic (for $0 < \alpha < 1$) as an element of the class $S(m,g)$ for $m = \langle\xi\rangle^2$ and $g_{x,\xi} = \langle x\rangle^{-2}|dx|^2 + \langle x\rangle^{2\alpha}\langle\xi\rangle^{-2}|d\xi|^2$. Moreover its spectrum is the real semi-axis $[0,\infty)$.

Our analysis will be based on the definition of complex powers of a non-negative operator due to Balakrishnan. As an application we will also study the semigroup generated by a non-negative pseudodifferential operator, that is its heat kernel.

Finally we briefly mention another application of our results to the so-called Schatten–von Neumann property for pseudodifferential operators (see, e.g., [11]). More precisely, we obtain the following *necessary and sufficient* conditions (see also [5]):

$$\Psi(m,g) \subset S_p\bigl(L^2\bigr) \quad \Longleftrightarrow \quad m \in L^p\bigl(\mathbb{R}^{2n}\bigr), \quad 1 \leqslant p < \infty,$$

where $\Psi(m,g)$ is the space of the closures in L^2 of the operators with Weyl symbol in $S(m,g)$. As usual, $S_p(L^2)$ denotes the class of compact operators in L^2 whose sequence of singular values is in ℓ^p.

2. Hypoelliptic symbols

We work in the context of Weyl–Hörmander calculus: refer to [15, Chapter XVIII] and [14] for further details.

We will employ the following notation. Given two functions $f, g : X \to \mathbb{R}$,

$$f(x) \prec g(x) \quad \forall x \in S \subset X,$$

means that there exists a constant $C > 0$ such that $f(x) \leqslant Cg(x)$ for all $x \in S$.

An *admissible metric* is a measurable function $g:(x,\xi) \mapsto g_{x,\xi}$, of $\mathbb{R}^n \times \mathbb{R}^n$ into the set of positive definite quadratic forms on $\mathbb{R}^n \times \mathbb{R}^n$, which is slowly varying, σ-temperate, and satisfies the uncertainty principle $h \leqslant 1$, where

$$h(x,\xi) = \left(\sup_{(t,\tau)} \frac{g_{x,\xi}(t,\tau)}{g^\sigma_{x,\xi}(t,\tau)}\right)^{1/2},$$

and g^σ is the *dual quadratic form*:

$$g^\sigma_{x,\xi}(t,\tau) = \sup_{g_{x,\xi}(y,\eta)=1} \sigma\big((t,\tau);(y,\eta)\big)^2,$$

with respect to the *standard symplectic form* $\sigma = \sum_{i=1}^n d\xi_i \wedge dx_i$ *in* $\mathbb{R}^n \times \mathbb{R}^n$.

A *g-weight* is a positive measurable function $m:\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, which is g-continuous and (σ,g)-temperate.

A smooth function $a:\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a *symbol* if there exists a g-weight m such that

$$\sup_{(x,\xi)} \frac{|a|^g_k(x,\xi)}{m(x,\xi)} < \infty \quad \text{for all } k \in \mathbb{N},$$

where $|a|^g_0(x,\xi) = |a(x,\xi)|$ and

$$|a|^g_k(x,\xi) = \sup_{(t_j,\tau_j)} \frac{|a^{(k)}((x,\xi);(t_1,\tau_1),\dots,(t_k,\tau_k))|}{g_{x,\xi}(t_1,\tau_1)^{1/2}\cdots g_{x,\xi}(t_k,\tau_k)^{1/2}} \quad \text{for } k \geqslant 1.$$

We denote by $S(m,g)$ the class of all symbols of weight m and metric g.

$S(m,g)$ is a Fréchet space with respect to the seminorms:

$$|a|_{k;S(m,g)} = \sup_{(x,\xi)} \frac{|a|^g_k(x,\xi)}{m(x,\xi)} \quad (k \in \mathbb{N}).$$

We set also $\|a\|_{k;S(m,g)} = \sup_{j\leqslant k} |a|_{j;S(m,g)}$ $(k \in \mathbb{N})$.

When $a \in S(m,g)$, we define the *pseudodifferential operator of Weyl symbol a as*

$$\mathcal{W}_a u(x) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a\left(\frac{1}{2}(x+y),\xi\right) u(y)\,dy\,d\xi.$$

$\mathcal{W}_a$ is a continuous operator on the Schwartz class $\mathcal{S}$, with a continuous extension to the tempered distributions $\mathcal{S}'$.

The following theorem is proved (see [15, Theorem 18.5.4]).

Theorem 2.1. *Given two symbols $a \in S(m,g)$ and $b \in S(p,g)$, we have that $\mathcal{W}_a\mathcal{W}_b$ is a pseudodifferential operator with symbol $a \# b \in S(mp,g)$ such that*

$$\mathcal{R}_N(a,b) = a \# b - \sum_{j=0}^N \frac{\{a,b\}_j}{(2i)^j j!} \in S\big(mph^{N+1},g\big) \tag{2.1}$$

for all $N \in \mathbb{N}$*, where* $\{a, b\}_0 = ab$*, and*

$$\{a, b\}_j = \left[\left(\sum_{i=1}^{n} \left(\frac{\partial}{\partial \xi_i} \frac{\partial}{\partial y_i} - \frac{\partial}{\partial x_i} \frac{\partial}{\partial \eta_i} \right) \right)^j a(x, \xi) b(y, \eta) \right]_{\substack{y=x \\ \eta=\xi}}$$

for $j > 0$.

More precisely, for each $N, k \in \mathbb{N}$ *there exists an integer* $l_{N,k}$ *such that*

$$\left\| \mathcal{R}_N(a, b) \right\|_{k; S(mph^{N+1}, g)} \prec \|a\|_{l_{N,k}; S(m,g)} \|b\|_{l_{N,k}; S(p,g)}$$

for all $a \in S(m, g)$ *and all* $b \in S(p, g)$.

The following is a generalization of the definition of hypoelliptic symbol given by Tulovskiĭ and Shubin, see [33] and [32, §25].

Definition 2.2. A symbol $a \in S(m, g)$ is g*-hypoelliptic* if there exists a positive constant R such that:

(i) for all $k \in \mathbb{N}$ we have[1]

$$|a|_k^g(x, \xi) \prec \left| a(x, \xi) \right| \quad \text{for } \left| (x, \xi) \right| \geqslant R; \tag{2.2}$$

(ii) there exists a g-weight m_0 such that

$$\left| a(x, \xi) \right| \succ m_0(x, \xi) \quad \text{for } \left| (x, \xi) \right| \geqslant R. \tag{2.3}$$

When $m_0 = m$ we say that the symbol a is g*-elliptic*.

We denote by $HS(m, m_0; g)$ the class of g-hypoelliptic symbols belonging to $S(m, g)$ and satisfying (2.3).

Remark. We do not require a to be slowly varying or temperate.

Now we prove a lemma we shall need later on.

Lemma 2.3. *Given two smooth functions* $a, b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$*, we have*

$$\left| \{a, b\}_j \right|_k^g \leqslant (2n)^j \sum_{l=0}^{k} \binom{k}{l} |a|_{j+l}^g |b|_{j+k-l}^g h^j$$

for all $k \in \mathbb{N}$*, and all* $j \in \mathbb{Z}_+$.

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[1] $|(x, \xi)|^2 = |x|^2 + |\xi|^2 = \sum_{i=1}^n x_i^2 + \sum_{i=1}^n \xi_i^2$.

Proof. *Fix* $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$. We know from [15, Lemma 18.6.4] that there exists a symplectic linear transformation χ and n positive real numbers $\lambda_1,\dots,\lambda_n$ such that

$$g_{x,\xi}\circ\chi(t,\tau)=\sum_{i=1}^{n}\lambda_i\big(t_i^2+\tau_i^2\big)\quad\forall(t,\tau).$$

In particular, by symplectic invariance of g^σ, we have $h(x,\xi)=\sup\lambda_i$.

Now one shows by induction that

$$\{a\circ\chi,b\circ\chi\}_j=\{a,b\}_j\circ\chi,$$

and

$$\{a,b\}_j=\sum_{|\alpha+\beta|=j}\frac{(-1)^\beta j!}{\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta a\partial_\xi^\beta\partial_x^\alpha b$$

for all $j\in\mathbb{Z}_+$.

Let $(x,\xi)=\chi(y,\eta)$; then we have

$$\begin{aligned}&\big|\{a,b\}\big|_j^{(k)}\big((x,\xi);\chi(t_1,\tau_1),\dots,\chi(t_n,\tau_n)\big)\\&\quad=\sum_{|\alpha+\beta|=j}\frac{(-1)^\beta j!}{\alpha!\beta!}\big(\partial_\eta^\alpha\partial_y^\beta(a\circ\chi)\partial_\eta^\beta\partial_y^\alpha(b\circ\chi)\big)^{(k)}\big((y,\eta);(t_1,\tau_1),\dots,(t_n,\tau_n)\big).\end{aligned}$$

Now we let $(x,\xi)=X$, $(y,\eta)=Y$, $(t_i,\tau_i)=T_i$, and we denote by $E_1,\dots,E_n,F_1,\dots,F_n$ the standard basis of $\mathbb{R}^n\times\mathbb{R}^n$. Then, thanks to Leibnitz formula for differentials, we have

$$\begin{aligned}&\big|\{a,b\}\big|_j^{(k)}\big(X;\chi(T_1),\dots,\chi(T_k)\big)\\&\quad=\sum_{|\alpha+\beta|=j}\frac{(-1)^\beta j!}{\alpha!\beta!}\sum_{i=0}^{k}\sum_{\substack{1\leqslant p_1<\dots<p_{k-i}\leqslant k\\1\leqslant q_1<\dots<q_i\leqslant k\\p_r\neq q_s,\,\forall r,s}}\big(\partial_Y^{(\alpha,\beta)}(a\circ\chi)\big)^{(k-i)}(Y;T_{p_1},\dots,T_{p_{k-i}})\\&\qquad\times\big(\partial_Y^{(\beta,\alpha)}(b\circ\chi)\big)^{(i)}(Y;T_{q_1},\dots,T_{q_i})\\&\quad=\sum_{|\alpha+\beta|=j}\frac{(-1)^\beta j!}{\alpha!\beta!}\sum_{i=0}^{k}\sum_{\substack{1\leqslant p_1<\dots<p_{k-i}\leqslant k\\1\leqslant q_1<\dots<q_i\leqslant k\\p_r\neq q_s,\,\forall r,s}}a^{(j+k-i)}\big(X;\underbrace{\chi(E_1),\dots}_{\beta_1},\dots,\underbrace{\chi(E_n),\dots}_{\beta_n},\dots,\\&\qquad\underbrace{\chi(F_1),\dots}_{\alpha_1},\dots,\underbrace{\chi(F_n),\dots}_{\alpha_n},\chi(T_{p_1}),\dots,\chi(T_{p_{k-i}})\big)\\&\qquad\times b^{(j+i)}\big(X;\underbrace{\chi(E_1),\dots}_{\alpha_1},\dots,\underbrace{\chi(E_n),\dots}_{\alpha_n},\dots,\underbrace{\chi(F_1),\dots}_{\beta_1},\dots,\underbrace{\chi(F_n),\dots}_{\beta_n},\\&\qquad\qquad\chi(T_{q_1}),\dots,\chi(T_{q_i})\big)\end{aligned}$$

$$\leqslant \sum_{i=0}^{k} \binom{k}{i} |a|^g_{j+k-i}(X)|b|^g_{j+i}(X) \sum_{|\alpha+\beta|=j} \frac{j!}{\alpha!\beta!} \lambda^\alpha \lambda^\beta g_X\big(\chi(T_1)\big)^{1/2} \cdots g_X\big(\chi(T_k)\big)^{1/2}$$

$$\leqslant (2n)^j \sum_{i=0}^{k} \binom{k}{i} |a|^g_{j+k-i}(X)|b|^g_{j+i}(X) h(X)^j g_X\big(\chi(T_1)\big)^{1/2} \cdots g_X\big(\chi(T_k)\big)^{1/2}. \qquad \square$$

Every pseudodifferential operator $\mathcal{W}_a$ is closable in L^2. We denote by $\overline{\mathcal{W}}_a$ its closure. When h is small enough, one can show that hypoelliptic operators have essentially only one closed extension.

Theorem 2.4. *Consider a g-hypoelliptic symbol $a \in HS(m, m_0; g)$ and assume there exists $N_0 \in \mathbb{N}$ such that*

$$h^{N_0} \prec \frac{\inf\{1, m_0\}}{m}. \tag{2.4}$$

Then the minimal and the maximal extension of $\mathcal{W}_a$ to L^2 coincide.

Proof. See [5, Theorem 2.5]. $\square$

3. Complex powers

We want to study complex powers of hypoelliptic pseudodifferential operators. So we begin by recalling some results on complex powers of a non-negative operator. Refer to [21] for proofs and more details.

Definition 3.1 *(Komatsu).* A closed operator A on a Banach space X is *non-negative* if:

(i) $(-\infty, 0)$ is contained in the resolvent set of A;
(ii) $\sup_{\lambda\in\mathbb{R}_+} \lambda\|(A+\lambda I)^{-1}\|_{\mathcal{B}(X)} < \infty$.

Set $\mathbb{C}_+ = \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\}$ and

$$\gamma_k(z) = \frac{\Gamma(k)}{\Gamma(z)\Gamma(k-z)} = \frac{(k-1)!\sin\pi z}{(k-1-z)\cdots(1-z)\pi} \tag{3.1}$$

for $k \in \mathbb{Z}_+$ and $z \in \mathbb{C} \setminus \mathbb{Z}$.

Proposition 3.2. *Consider a non-negative operator A on a Banach space X.*

Given $z \in \mathbb{C}_+$, and $u \in \mathcal{D}(A^{[\operatorname{Re} z]+1})$ we have that the integral

$$I^z_{A,k}u = \gamma_k(z) \int_0^\infty \lambda^{z-1}\big[A(A+\lambda I)^{-1}\big]^k u\, d\lambda$$

is convergent for all integers $k > \operatorname{Re} z$.

Moreover these integrals are independent of k:

$$I^z_{A,k+1}u = I^z_{A,k}u \quad \forall k > \operatorname{Re} z.$$

Proof. See [21, Proposition 3.1.3]. □

Definition 3.3 *(Balakhrishnan).* Given a non-negative operator A on a Banach space X and a complex number $z \in \mathbb{C}_+$, define a new operator J^z_A on X as

$$\begin{cases} \mathcal{D}(J^z_A) = \mathcal{D}(A^{[\operatorname{Re} z]+1}), \\ J^z_A u = I^z_{A,k}u \quad \text{for any } k > \operatorname{Re} z. \end{cases}$$

Theorem 3.4. *Assume that* A *is a non-negative, densely defined operator on a Banach space* X*, then*

$$A^z = \overline{J^z_A}, \quad z \in \mathbb{C}_+,$$

is the unique *family of operators which enjoys the following set of properties*:

(i) $\mathcal{D}(J^z_A) \subset \mathcal{D}(A^z)$;
(ii) A^z *is closed*;
(iii) $A^k = \underbrace{AA\cdots A}_{k\text{-times}}$, *with* $k \in \mathbb{Z}_+$;
(iv) $A^z A^w = A^{z+w}$;
(v) (*Spectral Mapping Theorem*) *The spectrum of* A^z *is given by*[2]

$$\sigma\big(A^z\big) = \big\{\lambda^z \colon \lambda \in \sigma(A)\big\};$$

(vi) *For all* $u \in \mathcal{D}(A^n)$*, with* $n \in \mathbb{Z}^+$*, the mapping*

$$z \mapsto A^z u$$

is analytic in the strip $\{z \in \mathbb{C} \colon 0 < \operatorname{Re} z < n\}$.

Proof. See [21, Theorems 3.1.5 and 3.1.8, Corollary 5.1.12 and Section 6.2]. □

Assume now that A is the closure in L^2 of a non-negative pseudodifferential operator $\mathcal{W}_a$. In next Theorem 3.6 we show that under suitable hypotheses $\overline{\mathcal{W}}^z_a$ is pseudodifferential.

Definition 3.5. We say that an admissible metric g satisfies the *strong uncertainty principle*, if there exists a positive constant δ such that

$$h(x,\xi) \prec \langle x,\xi\rangle^{-\delta} \quad \forall (x,\xi), \tag{3.2}$$

[2] The complex power λ^z is the principal branch $\lambda^z = \exp(z(\log|\lambda| + i \arg\lambda))$, with $-\pi < \arg\lambda \leqslant \pi$.

where

$$\langle x,\xi\rangle = \bigl(1+|x|^2+|\xi|^2\bigr)^{1/2}.$$

Theorem 3.6. *Consider an admissible metric g satisfying the strong uncertainty principle and a g-hypoelliptic symbol $a \in HS(m, m_0; g)$ such that*

$$\operatorname{Re} a(x,\xi) \geqslant -R\bigl|\operatorname{Im} a(x,\xi)\bigr| \tag{3.3}$$

for $|(x,\xi)| \geqslant R$, where R is a positive constant such that estimates (2.2) *and* (2.3) *are satisfied.*
Assume that $\overline{\mathcal{W}}_a$ is non-negative. *Let*

$$a_0 = \bigl(|a|\chi + 1 - \chi\bigr)e^{i(\arg a)\chi}, \tag{3.4}$$

where $\chi : \mathbb{R}^n \times \mathbb{R}^n \to [0,1]$ is a smooth function such that

$$\chi(x,\xi) = \begin{cases} 0, & \text{if } |(x,\xi)| \leqslant R, \\ 1, & \text{if } |(x,\xi)| \geqslant 2R. \end{cases}$$

Then for all $z \in \mathbb{C}_+$ there exists a g-hypoelliptic symbol

$$a^{\#z} \in S\bigl(m^{\operatorname{Re} z}, m_0^{\operatorname{Re} z}; g\bigr)$$

such that:

(i) *for all $k \in \mathbb{N}$ and $z \in \mathbb{C}_+$ we have*

$$\bigl|a^{\#z} - a_0^z\bigr|_k^g(x,\xi) \prec \bigl|a_0(x,\xi)\bigr|^{\operatorname{Re} z} h(x,\xi) \quad \forall (x,\xi) \tag{3.5}$$

(a_0^z *is the principal branch*);

(ii) *for all $z \in \mathbb{C}_+$ we have*

$$\overline{\mathcal{W}}_a^z = \overline{\mathcal{W}}_{a^{\#z}}.$$

Proof. We shall prove this theorem in Section 5. □

Remark. From Theorems 3.4 and 3.6 it follows that $a^{\#k} = \underbrace{a \# \cdots \# a}_{k\text{-times}}$, when $k \in \mathbb{Z}_+$.

We are also able to give a simple description of the domain of $\overline{\mathcal{W}}_a^z$:

Corollary 3.7. *Under the hypotheses of Theorem* 3.6, *we have that*

$$\mathcal{D}\bigl(\overline{\mathcal{W}}_a^z\bigr) = \bigl\{u \in L^2 \colon \widetilde{\mathcal{W}}_{a^{\#z}} u \in L^2\bigr\} \quad \text{for all } z \in \mathbb{C}_+,$$

where $\widetilde{\mathcal{W}}_{a^{\#z}}$ denotes the extension of $\mathcal{W}_{a^{\#z}}$ to the tempered distributions $\mathcal{S}'$.

Proof. Because $a^{\#z}$ is hypoelliptic, the result follows from Theorems 3.6 and 2.4. □

4. The symbol of the resolvent

We need several lemmas. The proof of the first lemma is elementary and is left to the reader.

Lemma 4.1. *Consider a smooth function $a_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$. Assume there exists a positive constant R such that*

$$\operatorname{Re} a_0(x,\xi) \geqslant -R\left|\operatorname{Im} a_0(x,\xi)\right| \quad \textit{for all } (x,\xi). \tag{4.1}$$

Then we have

$$\begin{aligned}\left|a_0(x,\xi)\right| &\leqslant \sqrt{1+R^2}\left|a_0(x,\xi)+\lambda\right|,\\ \lambda &\leqslant \sqrt{1+R^2}\left|a_0(x,\xi)+\lambda\right|\end{aligned} \tag{4.2}$$

for all (x,ξ) and $\lambda \geqslant 0$.

Lemma 4.2. *Consider an admissible metric g and a smooth function $a_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ satisfying* (4.1) *and such that for all $k \in \mathbb{Z}_+$*

$$|a_0|_k^g(x,\xi) \prec \left|a_0(x,\xi)\right| \quad \textit{for all } (x,\xi).$$

Then, for all $k \in \mathbb{Z}_+$ we have

$$\left|(a_0+\lambda)^{-1}\right|_k^g(x,\xi) \prec \frac{1}{|a_0(x,\xi)+\lambda|} \quad \textit{for all } (x,\xi) \textit{ and } \lambda > 0.$$

Proof. From (4.2) we have $|a_0+\lambda| > 0$ for all (x,ξ) and $\lambda > 0$. Then from estimate [15, (18.4.4)], for all $k \in \mathbb{Z}_+$ we have

$$\begin{aligned}\left|(a_0+\lambda)^{-1}\right|_k^g(x,\xi) &\prec \frac{1}{|a_0(x,\xi)+\lambda|}\left(\sum_{j=1}^{k}\left(\frac{|a_0+\lambda|_j^g(x,\xi)}{|a_0(x,\xi)+\lambda|}\right)^{1/j}\right)^k\\ &\prec \frac{1}{|a_0(x,\xi)+\lambda|}\left(\sum_{j=1}^{k}\left(\frac{|a_0(x,\xi)|}{|a_0(x,\xi)+\lambda|}\right)^{1/j}\right)^k\\ &\prec \frac{1}{|a_0(x,\xi)+\lambda|} \quad \text{for all } (x,\xi) \text{ and } \lambda > 0,\end{aligned}$$

thanks to Lemma 4.1. □

From the strong uncertainty principle and Lemma 4.1 one easily obtains the following lemma.

Lemma 4.3. *Consider an admissible metric g satisfying the strong uncertainty principle and a continuous function $a_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ satisfying* (4.1). *Assume there exist three positive constants ν, ν_0, R such that*

$$\langle x,\xi\rangle^{-\nu_0} \prec \left|a_0(x,\xi)\right| \prec \langle x,\xi\rangle^{\nu}. \tag{4.3}$$

Then we have

$$h(x,\xi)^{\nu_0/\delta}(1+\lambda) \prec \left|a_0(x,\xi)+\lambda\right| \prec h(x,\xi)^{-\nu/\delta}(1+\lambda) \tag{4.4}$$

for all (x,ξ) *and* $\lambda \geqslant 0$.

Lemma 4.4. *Consider an admissible metric g satisfying the strong uncertainty principle, two g-weights μ and θ, and a smooth function a_0 satisfying the hypotheses of Lemma* 4.3*. Assume that for all $\lambda \in \mathbb{R}_+$ two symbols ϕ_λ and ψ_λ are given, such that for all $k \in \mathbb{N}$ we have*

$$|\phi_\lambda|_k^g \prec |a_0|^{L'}|a_0+\lambda|^{M'}h^{N'}, \qquad |\psi_\lambda|_k^g \prec |a_0|^{L''}|a_0+\lambda|^{M''}h^{N''} \tag{4.5}$$

for all (x,ξ) *and* $\lambda > 0$, *with* $L', L'', M', M'', N', N'' \in \mathbb{R}$.

Then for all $N, k \in \mathbb{N}$ *we have*

$$|\phi_\lambda \# \psi_\lambda|_k^g \prec |a_0|^{L'+L''}|a_0+\lambda|^{M'+M''}h^{N'+N''}, \tag{4.6}$$

and

$$\left|\mathcal{R}_N(\phi_\lambda,\psi_\lambda)\right|_k^g \prec |a_0|^{L'+L''}|a_0+\lambda|^{M'+M''}h^{N'+N''+N+1} \tag{4.7}$$

for all (x,ξ) *and* $\lambda > 0$. *In* (4.7) $\mathcal{R}_N(\phi_\lambda,\psi_\lambda)$ *is defined in* (2.1).

Proof. From (4.5) and (4.4), for all $k \in \mathbb{N}$ we obtain

$$\|\phi_\lambda\|_{k;S(h^{\gamma'},g)} \prec (1+\lambda)^{M'} \quad \forall\lambda > 0,$$
$$\|\psi_\lambda\|_{k;S(h^{\gamma''},g)} \prec (1+\lambda)^{M''} \quad \forall\lambda > 0,$$

with[3]

$$\gamma' = \left(L'_- + M'_-\right)\nu_0/\delta - \left(L'_+ + M'_+\right)\nu/\delta + N',$$
$$\gamma'' = \left(L''_- + M''_-\right)\nu_0/\delta - \left(L''_+ + M''_+\right)\nu/\delta + N''.$$

From Lemma 2.3 and Theorem 2.1 for all $N_0, k \in \mathbb{N}$ there exists an integer $l_{N_0,k} \geqslant N_0 + k$ such that

$$\begin{aligned}
|\phi_\lambda \# \psi_\lambda|_k^g &\prec \|\phi_\lambda\|_{l_{N_0,k}}^g \|\psi_\lambda\|_{l_{N_0,k}}^g + \left|\mathcal{R}_{N_0}(\phi_\lambda,\psi_\lambda)\right|_k^g \\
&\prec \|\phi_\lambda\|_{l_{N_0,k}}^g \|\psi_\lambda\|_{l_{N_0,k}}^g + \left\|\mathcal{R}_{N_0}(\phi_\lambda,\psi_\lambda)\right\|_{k;S(h^{\gamma'+\gamma''+N_0+1},g)} h^{\gamma'+\gamma''+N_0+1} \\
&\prec \|\phi_\lambda\|_{l_{N_0,k}}^g \|\psi_\lambda\|_{l_{N_0,k}}^g + \|\phi_\lambda\|_{l_{N_0,k};S(h^{\gamma'},g)} \|\psi_\lambda\|_{l_{N_0,k};S(h^{\gamma''},g)} h^{\gamma'+\gamma''+N_0+1} \\
&\prec |a_0|^{L'+L''}|a_0+\lambda|^{M'+M''}h^{N'+N''} + (1+\lambda)^{M'+M''}h^{\gamma'+\gamma''+N_0+1}
\end{aligned}$$

for all (x,ξ) and $\lambda > 0$.

[3] As usual we set $x_- = \min\{x,0\}$ and $x_+ = \max\{x,0\}$.

Then (4.6) follows from (4.4), when we choose

$$\begin{aligned} N_0 &\geqslant \left(L'_+ + L''_+\right)\nu_0/\delta - \left(L'_- + L''_-\right)\nu/\delta \\ &\quad + \left(M'_+ + M''_+\right)\nu_0/\delta - \left(M'_- + M''_-\right)\nu/\delta + N' + N'' - \gamma' - \gamma'' - 1 \\ &= \left(|L'| + |L''| + |M'| + |M''|\right)(\nu_0 + \nu)/\delta - 1. \end{aligned}$$

The proof of (4.7) is similar. □

Theorem 4.5. *Consider an admissible metric g satisfying the strong uncertainty principle and a g-hypoelliptic symbol $a \in HS(m, m_0; g)$ satisfying* (3.3).

Then for each $\lambda \in \mathbb{R}_+$ there exists a g-hypoelliptic symbol

$$q_\lambda \in HS\left(m_0^{-1}, (m+\lambda)^{-1}; g\right),$$

such that:

(i) *for all $k \in \mathbb{N}$ we have*

$$\left| q_\lambda - \frac{1}{a_0 + \lambda} \right|_k^g \prec \frac{h}{|a_0 + \lambda|} \quad \forall (x, \xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+, \tag{4.8}$$

where a_0 is defined in (3.4);

(ii) *for all $k, N \in \mathbb{N}$ we have*

$$\begin{cases} |1 - q_\lambda \# (a+\lambda)|_k^g \prec h^N, \\ |1 - (a+\lambda) \# q_\lambda|_k^g \prec h^N \end{cases} \tag{4.9}$$

for all (x, ξ) and for $\lambda > 0$.

Proof. Thanks to temperance there exist two positive constant ν_0 and ν such that

$$m_0 \succ \langle x, \xi \rangle^{-\nu_0} \quad \text{and} \quad m \prec \langle x, \xi \rangle^{\nu} \quad \forall (x, \xi). \tag{4.10}$$

Then one checks that a_0 satisfies the hypotheses of Lemmas 4.1–4.3. Indeed, thanks to (3.3), $\arg a_0 = (\arg a)\chi$ is well defined and smooth. Moreover a_0 never vanishes and coincides with a when $|(x, \xi)| \geqslant 2R$. Finally, (4.1) is equivalent to

$$|\arg a_0| \leqslant \pi - \arctan \frac{1}{R},$$

which follows from (3.3) and the inequality $0 \leqslant \chi \leqslant 1$.

Define

$$\begin{aligned} r_{1,\lambda} &= 1 - (a_0 + \lambda)^{-1} \# (a + \lambda), \\ a_1 &= a - a_0. \end{aligned} \tag{4.11}$$

Because a_1 has compact support, for all $k \in \mathbb{N}$ we have

$$|a_1|_k^g \prec \langle x, \xi\rangle^{-\nu} h \prec |a_0| h \prec |a_0 + \lambda| h,$$
$$|a + \lambda|_k^g \prec |a_0 + \lambda| \tag{4.12}$$

for all (x, ξ) and $\lambda > 0$. Then from Lemma 4.4 we may conclude that for all $k \in \mathbb{N}$ we have

$$|r_{1,\lambda}|_k^g \prec \left|\frac{a_1}{a_0 + \lambda}\right| + \left|\mathcal{R}_0\big((a_0 + \lambda)^{-1}, a + \lambda\big)\right|_k^g \prec h \tag{4.13}$$

for all (x, ξ) and $\lambda > 0$ ($\mathcal{R}_0$ is defined in (2.1)).

Define $r_{0,\lambda} = 1$ and $r_{j,\lambda} = r_{1,\lambda} \# \cdots \# r_{1,\lambda}$ (j-times), when $j \geqslant 1$. Then from Lemma 4.4, for all $j, k \in \mathbb{N}$ we have

$$|r_{j,\lambda}|_k^g \prec h^j, \tag{4.14}$$

$$\left|r_{j,\lambda} \# (a_0 + \lambda)^{-1}\right|_k^g \prec \frac{h^j}{|a_0 + \lambda|} \tag{4.15}$$

for all (x, ξ) and $\lambda > 0$.

Now we want to carry out an asymptotic sum of the symbols $r_{j,\lambda} \# (a_0 + \lambda)^{-1}$. We make use of the following general result which is a simplified version of [28, Proposition 1.1.17].

Lemma 4.6. *Consider a decreasing sequence $X_0 \supset X_1 \supset \cdots \supset X_n \supset X_{n+1} \supset \cdots$, of Fréchet spaces such that for all $j \in \mathbb{N}$ the inclusion $X_{j+1} \subset X_j$ is continuous, and set $X = \bigcap_{n=0}^{\infty} X_n$.*

Assume that for all $j \in \mathbb{N}$ and all $s \in \mathbb{R}_+$ there exists a linear operator $L_{j,s} : X_j \to X_j$ which enjoys the following properties:

(i) *for all $j \in \mathbb{N}$ and all $c \in \mathbb{R}_+$ we have*

$$y - L_{j,s} y \in X \quad \textit{for all } y \in X_j;$$

(ii) *if $(p_{j,k})_{k \in \mathbb{N}}$ is a sequence of seminorms defining the topology of X_j, then for all $j, k \in \mathbb{N}$ there exists $\ell(j, k) \geqslant j, k$ such that*

$$\lim_{s \to \infty} p_{j,k}(L_{m,s} y) = 0$$

for all $m \geqslant \ell(j, k)$ and all $y \in X_m$.

Then, for every sequence $x_j \in X_j$ there exists a sequence of real numbers $s_j \in \mathbb{R}_+$ such that the series $\sum_{j=k}^{\infty} L_{j,s_j} x_j$ converges in X_k for all $k \in \mathbb{N}$. If we set $x = \sum_{j=0}^{\infty} L_{j,s_j} x_j \in X_0$, we have that

$$x - \sum_{j=0}^{n} x_j \in X_{n+1} \quad \textit{for all } n \in \mathbb{N}. \tag{4.16}$$

We write $x \sim \sum_{j=0}^{\infty} x_j$ to say that (4.16) is satisfied with $x \in X_0$ and $x_j \in X_j$. It is clear that x is uniquely determined by the sequence (x_j), up to an element in X.

Now we go back to the proof of our theorem. For each $j \in \mathbb{N}$, let X_j be the space of families of smooth functions $f_\lambda : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, $\lambda \in \mathbb{R}_+$, such that

$$p_{j,k}(f_\lambda) = \sup_{\substack{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n \\ \lambda\in\mathbb{R}_+}} \frac{|a_0(x,\xi)+\lambda||f_\lambda|_k^g(x,\xi)}{h(x,\xi)^j} < \infty$$

for all $k \in \mathbb{N}$.

Then define $L_s f_\lambda = \chi_s f_\lambda$, for $s \in \mathbb{R}_+$ where $\chi_s(x,\xi) = \chi(x/s, \xi/s)$. It is clear that $f_\lambda - L_s f_\lambda = (1-\chi_s) f_\lambda \in \mathcal{S} \subset X$, for all $f_\lambda \in X_0$ and all $s \in \mathbb{R}_+$.

Now it is not difficult to see that $\lim_{s\to\infty} p_{j,k}(\chi_s f_\lambda) = 0$ for all $j,k \in \mathbb{N}$ and $f \in X_l$, with $l > j$. Therefore, by Lemma 4.6 there exists $q_\lambda \in X_0$ such that

$$q_\lambda - \sum_{j=0}^{N} r_{j,\lambda} \# (a_0+\lambda)^{-1} \in X_{N+1} \quad \forall N \in \mathbb{N}. \tag{4.17}$$

Estimate (4.8) follows from (4.17) with $N = 0$. Moreover, for all $k \in \mathbb{N}$ we have

$$|q_\lambda|_k^g(x,\xi) \prec \frac{1}{|a_0(x,\xi)+\lambda|} \quad \text{for all } (x,\xi) \text{ and } \lambda > 0, \tag{4.18}$$

because $q_\lambda \in X_0$.

Because $h(x,\xi) \to 0$ as $|(x,\xi)| \to \infty$, (4.8) and (4.18) imply that $q_\lambda \in HS(m_0^{-1}, (m+\lambda)^{-1}; g)$.

It remains to prove estimates (4.9). We have

$$\begin{aligned}
&1 - q_\lambda \# (a+\lambda) \\
&\quad = 1 - \sum_{j=0}^{N} r_{j,\lambda} \# (a_0+\lambda)^{-1} \# (a+\lambda) - \mathcal{R}_N(q_\lambda, a+\lambda) \# (a+\lambda) \\
&\quad = 1 - \sum_{j=1}^{N} r_{1,\lambda} \# \cdots \# r_{1,\lambda} \# (1 - r_{1,\lambda}) - \mathcal{R}_N(q_\lambda, a+\lambda) \# (a+\lambda) \\
&\quad = r_{N+1,\lambda} - \mathcal{R}_N(q_\lambda, a+\lambda) \# (a+\lambda) \in X_{N+1},
\end{aligned}$$

which by Lemma 4.4 implies the first one of the estimates (4.9).

In order to prove the second one of the estimates (4.9), we observe that starting from

$$\tilde{r}_{1,\lambda} = 1 - (a+\lambda) \# (a_0+\lambda)^{-1}$$

and

$$\tilde{r}_{j,\lambda} = \underbrace{\tilde{r}_{1,\lambda} \# \cdots \# \tilde{r}_{1,\lambda}}_{j\text{-times}} \quad (j \geqslant 1),$$

we may consider

$$\tilde{q}_\lambda \sim \sum_{j=0}^{\infty} (a_0+\lambda)^{-1} \# \tilde{r}_{j,\lambda} \in X_0,$$

such that for all $k, N \in \mathbb{N}$ we have

$$\left|1-(a+\lambda) \# \tilde{q}_\lambda\right|_k^g \prec h^N \quad \text{for all } (x,\xi) \text{ and } \lambda > 0.$$

Thus, from Lemma 4.4, for all $k, N \in \mathbb{N}$ we have

$$|q_\lambda - \tilde{q}_\lambda|_k^g \prec \frac{h^N}{|a+\lambda|}$$

and therefore

$$\left|1-(a+\lambda) \# q_\lambda\right|_k^g \leqslant \left|1-(a+\lambda) \# \tilde{q}_\lambda\right|_k^g + \left|(a+\lambda) \# (q_\lambda - \tilde{q}_\lambda)\right|_k^g \prec h^N$$

for all (x,ξ) and $\lambda > 0$. $\square$

Theorem 4.7. *Under the hypotheses of Theorem* 3.6*, we have that for all $\lambda > 0$ the operator $\mathcal{W}_a + \lambda I : \mathcal{S} \to \mathcal{S}$ is invertible and $\mathcal{W}_a(\mathcal{W}_a + \lambda I)^{-1} : \mathcal{S} \to \mathcal{S}$ is a g-hypoelliptic pseudodifferential operator with Weyl symbol*

$$b_\lambda \in HS\big(m/m_0, m_0/(m+\lambda); g\big)$$

such that for all $k \in \mathbb{N}$ we have the estimates

$$\left|b_\lambda - \frac{a_0}{a_0+\lambda}\right|_k^g \prec \left|\frac{a_0 h}{a_0+\lambda}\right| \tag{4.19}$$

for $|(x,\xi)| \geqslant 2R$ and $\lambda > 0$.

Proof. We begin by observing that from Theorem 4.5 we have the following global hypoellipticity property: for all $\lambda > 0$ we have

$$u \in \mathcal{S}' \ \& \ (\widetilde{\mathcal{W}}_a + \lambda I)u \in \mathcal{S} \quad \Longrightarrow \quad u \in \mathcal{S},$$

where $\widetilde{\mathcal{W}}_a$ is the extension of $\mathcal{W}_a$ to $\mathcal{S}'$. Now we show that $\mathcal{W}_a + \lambda I : \mathcal{S} \to \mathcal{S}$, and its extension $\widetilde{\mathcal{W}}_a + \lambda I : \mathcal{S}' \to \mathcal{S}'$ to $\mathcal{S}'$ are invertible for all $\lambda > 0$.

In fact $\mathcal{W}_a + \lambda I$ is one-to-one because, by hypothesis, $\overline{\mathcal{W}}_a + \lambda I$ is. On the other hand we know that[4] $\mathcal{R}(\overline{\mathcal{W}}_a + \lambda I) = L^2$. Therefore, given any $f \in \mathcal{S} \subset L^2$ there exists $u \in L^2$ such that $(\overline{\mathcal{W}}_a + \lambda I)u = f$. But $u \in \mathcal{S}$ by hypoellipticity and therefore $(\mathcal{W}_a + \lambda I)u = f$, that is $\mathcal{W}_a + \lambda I$ is onto.

[4] We denote by $\mathcal{N}(A)$ and $\mathcal{R}(A)$ respectively the null space and the range of a linear operator A.

By transposition we have that $\widetilde{\mathcal{W}}_{\bar{a}} + \lambda I$ is one-to-one and onto. But, because a is g-hypoelliptic also $\bar{a}$ is g-hypoelliptic. It follows easily that $\mathcal{W}_{\bar{a}} + \lambda I$ is one-to-one and onto and therefore also $\widetilde{\mathcal{W}}_a + \lambda I$ is, by transposition.

Then for all $\lambda > 0$ we may consider $\mathcal{W}_a(\mathcal{W}_a + \lambda I)^{-1} : \mathcal{S} \to \mathcal{S}$, as well as its extension $\widetilde{\mathcal{W}}_a(\widetilde{\mathcal{W}}_a + \lambda I)^{-1} : \mathcal{S}' \to \mathcal{S}'$. We want to show that this operator is pseudodifferential. We have

$$\begin{aligned}\mathcal{W}_a(\mathcal{W}_a + \lambda I)^{-1} &= \mathcal{W}_a\mathcal{W}_{q_\lambda} + \mathcal{W}_{q_\lambda}\mathcal{W}_a\mathcal{W}_{1-(a+\lambda)\# q_\lambda} \\ &\quad + \mathcal{W}_{1-q_\lambda\#(a+\lambda)}\mathcal{W}_a(\mathcal{W}_a + \lambda I)^{-1}\mathcal{W}_{1-(a+\lambda)\# q_\lambda}. \end{aligned} \tag{4.20}$$

For all $r, s \in \mathbb{R}$, consider the weighted Sobolev spaces:

$$H^{r,s} = \{u \in \mathcal{S}' \colon \langle x\rangle^r \langle D\rangle^s u \in L^2\} = \{u \in \mathcal{S}' \colon \langle D\rangle^r \langle \xi\rangle^s \hat{u} \in L^2\}.$$

Between $H^{r,s}$ and $H^{-r,-s}$ there is a natural duality:

$$(u, v)_{L^2} \leqslant \|u\|_{H^{r,s}} \|v\|_{H^{-r,-s}} \quad \forall u, v \in \mathcal{S}. \tag{4.21}$$

Moreover we have

$$\mathcal{S} = \bigcap_{r,s} H^{r,s}, \qquad \mathcal{S}' = \bigcup_{r,s} H^{r,s},$$

with the topologies of $\mathcal{S}$ and $\mathcal{S}'$ equal to the initial and final topology of intersection and union. In particular it follows that an operator is continuous from $\mathcal{S}'$ into $\mathcal{S}$ if and only if it is continuous from $H^{r,s}$ into $H^{p,q}$ for all $r, s, p, q \in \mathbb{R}$.

From (4.9) and the strong uncertainty principle we obtain easily that for all $r, s, p, q \in \mathbb{R}$ we have

$$\|\mathcal{W}_{1-q_\lambda\#(a+\lambda)}u\|_{H^{r,s}} \prec \|u\|_{H^{p,q}}, \qquad \|\mathcal{W}_{1-(a+\lambda)\# q_\lambda}u\|_{H^{r,s}} \prec \|u\|_{H^{p,q}},$$
$$\|\mathcal{W}_{1-q_\lambda\#(a+\lambda)}\mathcal{W}_a u\|_{H^{r,s}} = \|\mathcal{W}_{(1-q_\lambda\#(a+\lambda))\# a}u\|_{H^{r,s}} \prec \|u\|_{H^{p,q}}$$

for all $u \in \mathcal{S}$ and $\lambda > 0$. Let

$$S_\lambda = \widetilde{\mathcal{W}}_{1-q_\lambda\#(a+\lambda)}\widetilde{\mathcal{W}}_a(\widetilde{\mathcal{W}}_a + \lambda I)^{-1}\widetilde{\mathcal{W}}_{1-(a+\lambda)\# q_\lambda}. \tag{4.22}$$

By hypothesis $\overline{\mathcal{W}}_a + \lambda I$ is non-negative, so we have

$$\sup_{\lambda>0}\|\overline{\mathcal{W}}_a(\overline{\mathcal{W}}_a + \lambda I)^{-1}\|_{\mathcal{B}(L^2)} \leqslant 1 + \sup_{\lambda>0}\|\lambda(\overline{\mathcal{W}}_a + \lambda I)^{-1}\|_{\mathcal{B}(L^2)} < \infty.$$

It follows that for all $r, s, p, q \in \mathbb{R}$ we have

$$\|S_\lambda u\|_{H^{r,s}} \prec \|u\|_{H^{p,q}} \quad \text{and} \quad \|S_\lambda u\|_{H^{r,s}} \prec \frac{1}{\lambda}\|u\|_{H^{p,q}},$$

and therefore

$$\|S_\lambda u\|_{H^{r,s}} \prec \min\left\{1, \frac{1}{\lambda}\right\}\|u\|_{H^{p,q}} \prec (1+\lambda)^{-1}\|u\|_{H^{p,q}} \tag{4.23}$$

for all $u \in \mathcal{S}$ and $\lambda > 0$.

These estimates imply that S_λ is a pseudodifferential operator with Weyl symbol σ_λ satisfying for all $\alpha, \beta \in \mathbb{N}^n$ and $M \in \mathbb{N}$ the estimate

$$\left|D_\xi^\alpha D_x^\beta \sigma_\lambda(x,\xi)\right| \prec \frac{\langle x,\xi\rangle^{-M}}{1+\lambda} \quad \text{for all } (x,\xi) \text{ and } \lambda > 0,$$

that is

$$|\sigma_\lambda|_k^g(x,\xi) \prec \frac{h(x,\xi)^N}{1+\lambda} \quad \text{for all } (x,\xi) \text{ and } \lambda > 0, \tag{4.24}$$

by the strong uncertainty principle.

From (4.20) and (4.22) we obtain that $\mathcal{W}_a(\mathcal{W}_a+\lambda)^{-1}$ is a pseudodifferential operator with Weyl symbol

$$b_\lambda = a \# q_\lambda + q_\lambda \# a - q_\lambda \# (a+\lambda) \# q_\lambda + \sigma_\lambda.$$

Then the result follows from Theorem 4.5, (4.11), (4.12), (4.24), and Lemmas 4.3 and 4.4. □

Corollary 4.8. *For all $\lambda > 0$ we have that $(\mathcal{W}_a + \lambda I)^{-1}$ is a g-hypoelliptic pseudodifferential operator with Weyl symbol*

$$\tilde{a}_\lambda = \frac{1}{\lambda}(1-b_\lambda).$$

Proof. We have

$$(\mathcal{W}_a+\lambda I)^{-1} = \frac{1}{\lambda}\big(I - \mathcal{W}_a(\mathcal{W}_a+\lambda I)^{-1}\big) = \frac{1}{\lambda}(I - \mathcal{W}_{b_\lambda}). \qquad \square$$

Remark. Because $a \# \tilde{a}_\lambda = b_\lambda$, for all $k \in \mathbb{N}$ we have the estimate:

$$\left|a \# \tilde{a}_\lambda - \frac{a_0}{a_0+\lambda}\right|_k^g \prec \left|\frac{a_0 h}{a_0+\lambda}\right| \tag{4.25}$$

for all (x,ξ) and $\lambda > 0$.

5. Proof of Theorem 3.6

In all of this section we assume that the hypotheses of Theorem 3.6 are satisfied. We set

$$\tilde{a}_\lambda^{\#0} = 1 \quad \text{and} \quad \tilde{a}_\lambda^{\#k} = \underbrace{\tilde{a}_\lambda \# \cdots \# \tilde{a}_\lambda}_{k\text{-times}} \quad (k \in \mathbb{Z}_+).$$

Lemma 5.1. *Given any symbol $q \in S(p,g)$, for all (x,ξ) and all $k \in \mathbb{N}$, the function*

$$\mathbb{R}_+ \ni \lambda \mapsto \big(q \# \tilde{a}_\lambda^{\#k}\big)(x,\xi) \tag{5.1}$$

is $\mathcal{C}^\infty$ *and*

$$\frac{\partial}{\partial\lambda}\big(q\,\#\,\tilde{a}_\lambda^{\#k}\big)(x,\xi) = -k\big(q\,\#\,\tilde{a}_\lambda^{\#(k+1)}\big)(x,\xi). \tag{5.2}$$

Proof. Because $\tilde{a}_\lambda$ is the symbol of the resolvent $(\mathcal{W}_a + \lambda I)^{-1}$, it must satisfy the resolvent identity

$$\tilde{a}_\lambda - \tilde{a}_{\lambda_0} = -(\lambda - \lambda_0)\tilde{a}_\lambda\,\#\,\tilde{a}_{\lambda_0},$$

from which, thanks to Theorem 4.7, Corollary 4.8 and Theorem 2.1, we obtain the estimate

$$\big|\tilde{a}_\lambda(x,\xi) - \tilde{a}_{\lambda_0}(x,\xi)\big| \prec \frac{|(\lambda-\lambda_0)|}{\lambda\lambda_0} \quad \forall\lambda > 0,$$

which implies that $\lambda \mapsto (q\,\#\,\tilde{a}_\lambda)(x,\xi)$ is continuous.

It follows that

$$\begin{aligned}&\frac{(q\,\#\,\tilde{a}_\lambda)(x,\xi) - (q\,\#\,\tilde{a}_{\lambda_0})(x,\xi)}{\lambda-\lambda_0}\\ &\quad = -(q\,\#\,\tilde{a}_\lambda\,\#\,\tilde{a}_{\lambda_0})(x,\xi) \to -(q\,\#\,\tilde{a}_{\lambda_0}\,\#\,\tilde{a}_{\lambda_0})(x,\xi)\end{aligned}$$

as $\lambda \to \lambda_0$. This proves (5.2) for $k = 1$. The case corresponding to $k > 1$ follows by induction. Identity (5.2) implies also that (5.1) is $\mathcal{C}^\infty$. □

Lemma 5.2. *Given* $z \in \mathbb{C}_+$, *and* $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, *we have that the integral*

$$p_{a,z,k}(x,\xi) = \gamma_k(z)\int_0^\infty \lambda^{z-1}(a\,\#\,\tilde{a}_\lambda)^{\#k}(x,\xi)\,d\lambda$$

is convergent for all integers $k > \operatorname{Re} z$.

Moreover, for all integers $k > \operatorname{Re} z$ *we have*

$$p_{a,z,k}(x,\xi) = p_{a,z,k+1}(x,\xi). \tag{5.3}$$

Proof. From (4.25) and Lemma 4.4 for all $k, l \in \mathbb{N}$ we obtain the estimate:

$$\big|(a\,\#\,\tilde{a}_\lambda)^{\#k}\big|_l^g(x,\xi) \prec \left|\frac{a_0(x,\xi)}{a_0(x,\xi)+\lambda}\right|^k$$

for all (x,ξ) and $\lambda > 0$. This implies (pointwise) integrability. So we have only to prove (5.3).

Because $\tilde{a}_\lambda$ is the symbol of the resolvent $(\mathcal{W}_a + \lambda I)^{-1}$, a and $\tilde{a}_\lambda$ commute: $a\,\#\,\tilde{a}_\lambda = \tilde{a}_\lambda\,\#\,a$. Therefore, thanks to Lemma 5.1, an integration by parts gives

$$\int_0^\infty \lambda^{z-1}(a \# \tilde{a}_\lambda)^{\#k}\,d\lambda = \int_0^\infty \lambda^{z-1}\bigl(a^{\#k} \# \tilde{a}_\lambda^{\#k}\bigr)\,d\lambda$$

$$= \frac{1}{z}\bigl[\lambda^z\bigl(a^{\#k} \# \tilde{a}_\lambda^{\#k}\bigr)\bigr]_{\lambda=0}^{\lambda=\infty} + \frac{k}{z}\int_0^\infty \lambda^z\bigl(a^{\#k} \# \tilde{a}_\lambda^{\#(k+1)}\bigr)\,d\lambda$$

$$= \frac{k}{z}\int_0^\infty \lambda^{z-1}(a \# \tilde{a}_\lambda)^{\#k}\,d\lambda - \frac{k}{z}\int_0^\infty \lambda^{z-1}(a \# \tilde{a}_\lambda)^{\#(k+1)}\,d\lambda,$$

because

$$\lambda\bigl(a^{\#k} \# \tilde{a}_\lambda^{\#(k+1)}\bigr) + \bigl(a^{\#(k+1)} \# \tilde{a}_\lambda^{\#(k+1)}\bigr) = (\lambda + a) \# \tilde{a}_\lambda \# \bigl(a^{\#k} \# \tilde{a}_\lambda^{\#k}\bigr)$$
$$= (a \# \tilde{a}_\lambda)^{\#k}.$$

It follows that

$$\int_0^\infty \lambda^{z-1}(a \# \tilde{a}_\lambda)^{\#k}\,d\lambda = \frac{k}{k-z}\int_0^\infty \lambda^{z-1}(a \# \tilde{a}_\lambda)^{\#(k+1)}\,d\lambda$$

which implies (5.3) because from (3.1) we have $\gamma_k(z)k/(k-z) = \gamma_{k+1}(z)$. $\square$

Thanks to Lemma 5.2, for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $z \in \mathbb{C}_+$, we may define

$$a^{\#z}(x,\xi) = p_{a,z,k}(x,\xi),$$

where k is any integer greater than $\operatorname{Re} z$.

Now we show that for all $l \in \mathbb{N}$ and $z \in \mathbb{C}_+$ we have

$$\bigl|a^{\#z} - a_0^z\bigr|_l^g(x,\xi) \prec \bigl|a_0(x,\xi)\bigr|^{\operatorname{Re} z} h(x,\xi) \quad \forall (x,\xi). \tag{5.4}$$

Because $h(x,\xi) \to 0$ as $|(x,\xi)| \to \infty$, (5.4) implies in particular that

$$a^{\#z} \in S\bigl(m^{\operatorname{Re} z}, m_0^{\operatorname{Re} z}; g\bigr).$$

From (4.25), Lemmas 4.4 and 4.1 for all $k, l \in \mathbb{N}$ we obtain the estimate

$$\left|(a \# \tilde{a}_\lambda)^{\#k} - \left(\frac{a_0}{a_0 + \lambda}\right)^k\right|_l^g(x,\xi) \prec \left(\frac{|a_0(x,\xi)|}{|a_0(x,\xi)| + \lambda}\right)^k h(x,\xi) \tag{5.5}$$

for all (x,ξ) and $\lambda > 0$.

On the other side, by using the identity (see e.g. [10, 3.194.3, p. 285])

$$\int_0^\infty \frac{\lambda^{z-1} w^k}{(w+\lambda)^k}\,d\lambda = \frac{w^z}{\gamma_k(z)},$$

with $k \in \mathbb{Z}_+$, $z \in \mathbb{C}_+$ and $w \in \mathbb{C} \setminus \{0\}$ such that $\operatorname{Re} z < k$ and $|\arg w| < \pi$, we obtain

$$a^{\#z} - a_0^z = \gamma_k(z) \int_0^\infty \lambda^{z-1} \left[(a \,\#\, \tilde{a}_\lambda)^{\#k} - \left(\frac{a_0}{a_0 + \lambda} \right)^k \right] d\lambda.$$

So from estimate (5.5) we have

$$\left| a^{\#z} - a_0^z \right|_l^g \prec \left| \gamma_k(z) \right| \int_0^\infty \lambda^{\operatorname{Re} z - 1} \left(\frac{|a_0|}{|a_0| + \lambda} \right)^k h \, d\lambda = \frac{|\gamma_k(z)|}{\gamma_k(\operatorname{Re} z)} |a_0|^{\operatorname{Re} z} h,$$

which is (5.4).

In order to finish the proof of Theorem 3.6 we have show that

$$(\overline{\mathcal{W}}_a)^z = \overline{\mathcal{W}}_{a^{\#z}} \quad \forall z \in \mathbb{C}_+.$$

It suffices to prove this identity on $\mathcal{S}$. But when $u \in \mathcal{S}$ we have

$$(\overline{\mathcal{W}}_a)^z u = J^z_{\overline{\mathcal{W}}_a} u = \gamma_k(z) \int_0^\infty \lambda^{z-1} \big(\mathcal{W}_a (\mathcal{W}_a + \lambda I)^{-1} \big)^k u \, d\lambda$$

for $k > \operatorname{Re} z > 0$. On the other hand,

$$\big(\mathcal{W}_a (\mathcal{W}_a + \lambda I)^{-1} \big)^k = \mathcal{W}_{(a \# \tilde{a}_\lambda)^{\#k}},$$

so we have only to show that

$$\mathcal{W}_{a^{\#z}} u = \gamma_k(z) \int_0^\infty \lambda^{z-1} \mathcal{W}_{(a \# \tilde{a}_\lambda)^{\#k}} u \, d\lambda \tag{5.6}$$

for all $k > \operatorname{Re} z > 0$ and all $u \in \mathcal{S}$.

We need the following lemma.

Lemma 5.3. *For all $\lambda \in \mathbb{R}_+$, consider a symbol $\phi_\lambda \in S(\mu, g)$. Assume that*

(i) $\mathbb{R}_+ \ni \lambda \mapsto \phi_\lambda(x, \xi)$ *is continuous for all* (x, ξ);
(ii) $\mathbb{R}_+ \ni \lambda \mapsto |\phi_\lambda|_{k; S(\mu, g)}$ *is integrable on* $\mathbb{R}_+$ *for all* $k \in \mathbb{N}$.

Then

$$\psi(x, \xi) = \int_0^\infty \phi_\lambda(x, \xi) \, d\lambda$$

exists and belongs to $S(\mu, g)$ *and*

$$\mathcal{W}_{\psi} u = \int_{0}^{\infty} \mathcal{W}_{\phi_\lambda} u \, d\lambda \tag{5.7}$$

for all $u \in \mathcal{S}$.

Proof. If we let $k = 0$ in hypothesis (ii), we have that $\mathbb{R} \ni \lambda \mapsto \phi_\lambda(x, \xi)$ is integrable. By hypothesis (i), $\psi(x, \xi)$ is the limit of a sequence of Riemann sums $\sum_{j=1}^{J} \phi_{\lambda_j}(x, \xi)\Delta\lambda_j$.

By hypothesis (ii), these Riemann sums are bounded in the symbol space $S(\mu, g)$. It follows that $\psi \in S(\mu, g)$ and that $\mathcal{W}_\psi u(x)$ is the limit of the Riemann sums $\sum_{j=1}^{J} \mathcal{W}_{\phi_{\lambda_j}} u(x)\Delta\lambda_j$, thanks to [15, Theorem 18.6.2]. Because these Riemann sums converge also to $\int_0^\infty \mathcal{W}_{\phi_\lambda} u(x)\, d\lambda$, we obtain (5.7). □

Turning back to the proof of Theorem 3.6, we remark that

$$\lambda \mapsto \lambda^{z-1}(a \# \tilde{a}_\lambda)^{\# k}(x, \xi)$$

is continuous with respect to λ, so it remains to show that there exists a symbol μ such that the seminorms

$$\left|\lambda^{z-1}(a \# \tilde{a}_\lambda)^{\# k}\right|_{l; S(\mu, g)}$$

are integrable with respect to λ over $(0, \infty)$, for $k > \operatorname{Re} z$ and $l \in \mathbb{N}$.

Now for all $k, l \in \mathbb{N}$ we have the estimates

$$\left|(a \# \tilde{a}_\lambda)^{\# k}\right|_l^g (x, \xi) \prec \left|\frac{a_0(x, \xi)}{a_0(x, \xi) + \lambda}\right|^k$$

for all (x, ξ) and $\lambda > 0$. So from Lemma 4.3 we obtain

$$\sup_{\lambda > 0} \sup_{(x, \xi)} \frac{|(1 + \lambda)^k (a \# \tilde{a}_\lambda)^{\# k}|_l^g (x, \xi)}{h(x, \xi)^{-(\nu + \nu_0)/\delta}} < \infty \quad \forall k, l \in \mathbb{N},$$

that is, for all $k, l \in \mathbb{N}$ we have

$$\left|\lambda^{z-1}(a \# \tilde{a}_\lambda)^{\# k}\right|_{l; S(h^{-(\nu+\nu_0)/\delta}, g)} \prec \frac{\lambda^{\operatorname{Re} z - 1}}{(1 + \lambda)^k} \quad \forall \lambda > 0,$$

which is integrable for $k > \operatorname{Re} z$. □

6. Semigroup of the square-root of a non-negative operator

Square-roots of non-negative operators are the generators of analytic semigroups, as stated by the following theorem.

Theorem 6.1. *Consider a non-negative operator A with dense domain on a Banach space X. Then $-A^{1/2}$ is the infinitesimal generator of an analytic semigroup $e^{-tA^{1/2}}$ of amplitude less than $\pi/2$ and such that*

$$e^{-tA^{1/2}} = \frac{2}{\pi} \lim_{\Lambda\to\infty} \int_0^{\Lambda} \lambda \sin(t\lambda)\bigl(A+\lambda^2\bigr)^{-1}\, d\lambda \quad \text{for } t>0 \tag{6.1}$$

where the limit is taken in $\mathcal{B}(X)$.

Proof. See [21, Theorems 5.5.2, 5.4.1, and A.7.6]. □

If A is the closure in L^2 of a pseudodifferential operator with symbol $a \in HS(m, m_0; g)$ satisfying the hypotheses of Theorem 3.6, then we know that the infinitesimal generator of $e^{-tA^{1/2}}$ is the closure of the pseudodifferential operator with symbol $a^{\#1/2} \in S(m^{1/2}, m_0^{1/2}; g)$ satisfying (3.5) with $z = 1/2$. In the next theorem we show that also $e^{-tA^{1/2}}$ is pseudodifferential.

Theorem 6.2. *Consider a symbol $a \in H(m, m_0; g)$ satisfying the hypotheses of Theorem* 3.6. *Then there exists a semigroup of symbols*

$$\sigma_t = e^{\#(-ta^{\#1/2})} \in S(1; g),$$

such that

$$e^{-tA^{1/2}} = \overline{\mathcal{W}}_{\sigma_t} \quad \text{for all } t \geqslant 0.$$

Moreover the symbol σ_t is smooth with respect to t, x and ξ and for all $k, N \in \mathbb{N}$ and $T > 0$ satisfies the estimates

$$t^N \bigl|\partial_t^l \sigma_t\bigr|_k^g \prec |a_0|^{l/2-N/2} \quad \text{for all } t \in [0,T] \text{ and } (x,\xi) \in \mathbb{R}^{2n}, \tag{6.2}$$

where a_0 is given by (3.4).

Proof. Let $B = \rho(x, D) = A^{1/2}$. From Theorem 3.3 we know that $\rho \in S(m^{1/2}, m_0^{1/2}, g)$. Moreover we can assume that the estimates satisfied by ρ as a g-hypoelliptic symbol hold for every $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Similarly we can suppose that for some constant $R > 0$, ρ satisfies the estimate

$$\operatorname{Re}\rho(x,\xi) \geqslant R\bigl|\operatorname{Im}\rho(x,\xi)\bigr| \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{6.3}$$

We start by constructing the so-called heat parametrix, namely an operator $U(t) = u(t, x, D_x)$, with a smooth symbol $u(t,\cdot) \in S(1, g)$, satisfying

$$\begin{cases} (\partial_t + \rho(x, D))U(t) = K(t), \\ U(0) = I \end{cases} \tag{6.4}$$

for some operator K having kernel in $\mathcal{C}^\infty([0,\infty), \mathcal{S}(\mathbb{R}^{2n}))$; by using (6.1) we show that the operator e^{-tB} is pseudodifferential for all $t \geqslant 0$. Finally we compare $U(t)$ and e^{-tB}.

We look for u in the form $u \sim \sum_{j=0}^{\infty} u_j$ with $\partial_t^l u_j \in S(m^{l/2}h^j, g)$, for all $l \geqslant 0$. Then from (6.4) we obtain the following transport equations in $S(mh^j, g)$:

$$\begin{cases} \partial_t u_j + \sum_{k+l=j} \frac{\{\rho, u_k\}_l}{(2i)^l l!} = 0, \\ u_0(0,x,\xi) = 1, & \text{if } j > 0. \\ u_j(0,x,\xi) = 0, \end{cases} \tag{6.5}$$

For $j = 0$ we obtain $u_0 = e^{-t\rho}$. For $j > 0$, we have to solve the equations

$$\begin{cases} \partial_t u_j + \rho u_j + \sum_{\substack{k+l=j \\ k<j}} \frac{\{\rho, u_k\}_l}{(2i)^l l!} = 0, \\ u_j(0,x,\xi) = 0, \end{cases} \tag{6.6}$$

from which one easily verifies by induction that u_j can be written as $u_j = e^{-t\rho} p_j$ where $p_j(t,x,\xi)$ satisfies the estimates

$$\left|\partial_t^l p_j\right|_q^g \leqslant C \sum_{s=1}^{q+2j} t^s |\rho|^{s+l} h^j. \tag{6.7}$$

From these estimates and (6.3) it follows that

$$t^N \left|\partial_t^l u_j\right|_k^g \prec |\rho|^{l-N} h^j \quad \text{for all } (x,\xi) \text{ and } t \geqslant 0.$$

Then from Lemma 4.6 there exists $u \in S(1, g)$ satisfying (6.4) such that for all $l, k, N \in \mathbb{N}$

$$t^N \left|\partial_t^l u\right|_k^g \prec |a_0|^{l/2 - N/2} \quad \text{for all } (x,\xi) \text{ and } t \geqslant 0.$$

Since the operator e^{-tB} solves (6.4) with $K = 0$ we know that

$$U(t) - e^{-tB} = \int_0^t e^{-(t-s)B} K(s)\, ds. \tag{6.8}$$

Therefore, in order to obtain the estimates (6.2), we need to show that the right-hand side of (6.8) is an operator with a kernel in $\mathcal{C}^\infty([0,\infty), \mathcal{S}(\mathbb{R}^{2n}))$. This will be accomplished once we verify that $e^{-tB} \in \mathcal{C}^\infty([0,\infty), \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)))$.

To this end, we start by showing that the operator e^{-tB} is a pseudodifferential operator with a symbol

$$\sigma \in \mathcal{C}^0\big([0,\infty), S(1,g)\big). \tag{6.9}$$

This follows from the representation formula (6.1) and Theorem 4.7. Indeed, we have

$$e^{-tB} = \lim_{\Lambda\to\infty} \frac{2}{\pi}\int_0^{\Lambda} \frac{\sin(t\lambda)}{\lambda} I\, d\lambda - \lim_{\Lambda\to\infty} \frac{2}{\pi}\int_0^{\Lambda} \frac{\sin(t\lambda)}{\lambda} A\bigl(A+\lambda^2\bigr)^{-1}\, d\lambda$$

$$= I - \lim_{\Lambda\to\infty} \frac{2}{\pi}\int_0^{\Lambda} \frac{\sin(t\lambda)}{\lambda} A\bigl(A+\lambda^2\bigr)^{-1}\, d\lambda.$$

From Theorem 4.7 we have that $A(A+\lambda^2)^{-1}$ is a pseudodifferential operator with symbol b_{λ^2} satisfying (4.19). Choose $0 < \epsilon < \min\{1, \delta/\nu\}$, with the constant δ and ν given in (3.2) and (4.10), respectively. Then we have

$$|b_{\lambda^2}|_k^g \prec \left|\frac{a_0}{a_0+\lambda^2}\right|^{1-\epsilon} h^{1-\epsilon\nu/\delta} \left|\frac{1}{a_0+\lambda^2}\right|^{\epsilon} \prec \frac{1}{\lambda^{2\epsilon}}.$$

It follows that

$$\left|\frac{\sin(t\lambda)}{\lambda} b_{\lambda^2}\right|^g_{k;S(h^{1-\epsilon\nu/\delta},g)} \prec \min\bigl\{t, \lambda^{-1-2\epsilon}\bigr\},$$

which gives (6.9) in view of Lemma 5.3.

Now we know that $\frac{d}{dt}e^{-tB}\phi = -Be^{-tB}\phi$ for every $t \geqslant 0$ and every $\phi \in \mathcal{S}$. Hence $\sigma_t = -\rho \,\#\, \int_0^t \sigma_s\, ds + 1$, from which it follows by induction that $\sigma \in \mathcal{C}^k([0,\infty), S(m^{k/2}+1, g))$ for every $k \geqslant 0$. Therefore $e^{-tB} \in \mathcal{C}^\infty([0,\infty), \mathcal{L}(\mathcal{S},\mathcal{S}))$ and the proof is complete. $\square$

Eventually we mention a possible application of this result. Precisely, consider the following boundary value problem:

$$\begin{cases} u''(t) - Au(t) = 0, \\ u(0) = \phi \end{cases} \tag{6.10}$$

in $[0,\infty) \times \mathbb{R}^n$, where A is the closure in L^2 of a pseudodifferential operator $\mathcal{W}_a$ satisfying the hypotheses of Theorem 3.6 and $\phi \in L^2$. It is well known (see for example [21, Theorem 6.3.2]) that under these assumptions the unique bounded solution of (6.10) is given by $u(t) = S(t)\phi$, with $S(t) = e^{-tA^{1/2}}$, $t \geqslant 0$.

By resorting to the Sobolev-type spaces introduced by Bony and Chemin in [4], and wave front sets like those considered by Melrose [23] or Coriasco and Maniccia [6], Theorem 6.2 could then be used to obtain regularity results *of global nature* for the solution $u(t)$. However we do not do this here.

7. Schatten–von Neumann classes

Another interesting application of Theorem 3.6 is a very simple characterization of pseudodifferential operators which belong to Schatten–von Neumann classes. We briefly present a result of ours in this connection, referring to [5] for complete proofs.

A compact operator A in a Hilbert space X is in the *Schatten–von Neumann class* $S_p(X)$, with $1 \leqslant p < \infty$, if the sequence of its *singular values* $s_j(A)$ is in ℓ^p. (The singular values of A are the eigenvalues of $|A| = (A^*A)^{1/2}$.)

$S_p(X)$ is a Banach algebra with respect to the norm

$$\|A\|_{S_p} = \left\{ \sum_{j=1}^{\infty} s_j(A)^p \right\}^{1/p}.$$

The elements of $S_2(X)$ are the *Hilbert–Schmidt operators*, while $S_1(X)$ is the algebra of *trace class operators*.

Theorem 7.1. *Consider an admissible metric g satisfying the strong uncertainty principle. For any g-weight m and all $1 \leqslant p < \infty$, we have*

$$m \in L^p(\mathbb{R}^{2n}) \quad \Longleftrightarrow \quad \Psi(m, g) \subset S_p(L^2), \tag{7.1}$$

where

$$\Psi(m, g) = \{\overline{\mathcal{W}}_a \colon a \in S(m, g)\}.$$

Remark. When $p = 1$ the implication "$\Rightarrow$" in (7.1) follows from a result of Hörmander [13, Theorem 3.9]. For general p, it should be also possible to prove the equivalence (7.1) under rather general hypotheses by non-trivial interpolation arguments, as pointed out to us by Toft.

Proof. We need the following lemma.

Lemma 7.2. *Consider an admissible metric g satisfying the strong uncertainty principle. Then for all g-hypoelliptic symbols a we have*

$$\overline{\mathcal{W}}_a \in S_p(L^2) \quad \Longleftrightarrow \quad a \in L^p(\mathbb{R}^{2n}).$$

Proof. (See [5, Proposition 3.1] for details.) The operator with symbol $\bar{a} \# a$ is non-negative, hence we have

$$\begin{aligned} \overline{\mathcal{W}}_a \in S_p(L^2) \quad &\Longleftrightarrow \quad |\overline{\mathcal{W}}_a|^{p/2} \in S_2(L^2) \\ &\Longleftrightarrow \quad (\bar{a} \# a)^{\# p/4} \in L^2(\mathbb{R}^{2n}) \quad \Longleftrightarrow \quad a \in L^p(\mathbb{R}^{2n}). \qquad \square \end{aligned}$$

Suppose that $\Psi(m, g) \subset S_p(L^2)$. Without loss of generality we may assume that the weight m is a symbol in its own class (see [15, p. 143]). Then m belongs to L^p, by Lemma 7.2.

Assume now that $m \in L^p$ and let $a \in S(m, g)$. By linearity we may assume that a is real and non-negative. Thus $m + a$ is g-elliptic and therefore by Lemma 7.2

$$\overline{\mathcal{W}}_a = \overline{\mathcal{W}}_{m+a} - \overline{\mathcal{W}}_a \in S_p(L^2). \qquad \square$$

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